MATH1010 Assignment 3

Suggested Solution

1. Let $x \in (a, b]$

   By mean value theorem,$f(x) - f(a) = f'(c)(x - a)$ for some $c \in (a, x)$

   Since $f'(x) = 0$ for any $x \in (a, b)$, we have $f(x) = f(a)$ for any $x \in (a, b]$

   Hence f is a constant function.

2. For $0 < y < x$ and $p > 1$

   Let $f(z) = z^p$

   By mean value theorem,

$$
\begin{aligned}
f(x) - f(y) &= f'(c)(x - y) \quad \text{for some} \quad c \in (y, x) \\
&= pc^{p-1}(x - y)
\end{aligned}
$$

   Thus $py^{p-1}(x - y) \leqslant pc^{p-1}(x - y) \leqslant px^{p-1}(x - y)$

   And $py^{p-1}(x - y) \leqslant x^p - y^p \leqslant px^{p-1}(x - y)$

3. For $0 \leqslant x_1 < x_2 < x_3 \leqslant \pi$,

$$
\sin x_2 - \sin x_1 = \cos(a)(x_2 - x_1) \quad \text{for some} \quad a \in (x_1, x_2)
$$

$$
\sin x_3 - \sin x_2 = \cos(b)(x_3 - x_2) \quad \text{for some} \quad b \in (x_2, x_3)
$$

   Then we have $0 < a < b < \pi$ and $\cos(x)$ is strictly decreasing function on $(0, \pi)$

   Hence $\cos(a) > \cos(b)$ and $\frac{\sin x_2 - \sin x_1}{x_2 - x_1} = \cos(a) > \cos(b) = \frac{\sin x_3 - \sin x_2}{x_3 - x_2}$

4. Let $f(x) = \ln(1+x)$, by mean value theorem

$$f(x) - f(0) = f'(c)(x-c) \quad \text{for some} \quad c \in (0,x)$$

$$\begin{aligned} \ln(1+x) &= \frac{1}{1+c}x \\ \frac{1}{1+x} &< \frac{1}{1+c} < 1 \\ \frac{x}{1+x} &< \frac{x}{1+c} < x \end{aligned}$$

Therefore, $\frac{x}{1+x} < \ln(1+x) < x$ for $x > 0$.

Putting $x = \frac{1}{y}$, then we get $\frac{y}{1+y} < \ln(1+\frac{1}{y}) < \frac{1}{y}$

5. Let $f(x) = \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4 + \ldots + \frac{a_n}{n+1}x^{n+1}$

   $f'(x) = a_1x + a_2x^2 + \ldots + a_nx^n$

   By mean value theorem, $f(1) - f(0) = f'(c)(1-0)$ for some $c \in (0,1)$

   Hence $\frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} + \ldots + \frac{a_n}{n+1} = a_1c + a_2c^2 + \ldots + a_nc^n$

   So $a_1x + a_2x^2 + \ldots + a_nx^n = \frac{a_1}{2} + \frac{a_2}{3} + \ldots + \frac{a_n}{n+1}$ has root between 0 and 1

6. For $a \neq 0$,

   By mean value theorem, $f(a+b) - f(b) = f'(h)a \quad \text{for some} \quad h \in (b, a+b)$

   $f(a) - f(0) = f'(k)a \quad \text{for some} \quad k \in (0,a)$

   Then we have $h \geqslant k$ and $f'(x)$ is monotonic increasing on$(0,\infty)$

Therefore,

$$\begin{aligned} f(a+b)-f(b) &= f'(h)a \\ &\geqslant f'(k)a \\ &= f(a) \\ f(a+b) &\geqslant f(a)+f(b) \end{aligned}$$

For $a=0$,L.H.S= $f(b)$ =R.H.S

7. For $x, y \in \mathbb{R}$,

$$\begin{aligned} \sin x-\sin y &= \cos(c)(x-y) && \text{for } c \text{ between } x \text{ and } y \\ |\sin x-\sin y| &= |cos(c)||x-y| \\ &\leqslant (1)(|x-y|) \\ &= |x-y| \end{aligned}$$

So $\sin x$ satisfies the Lipschitz condition on $\mathbb{R}$